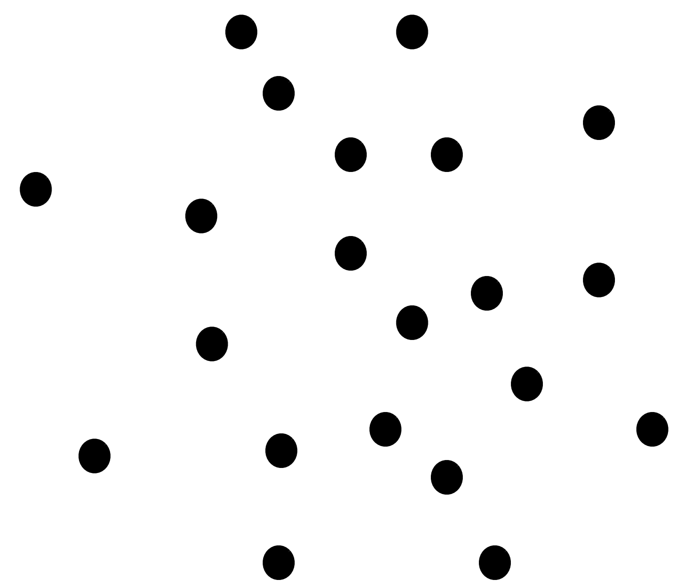
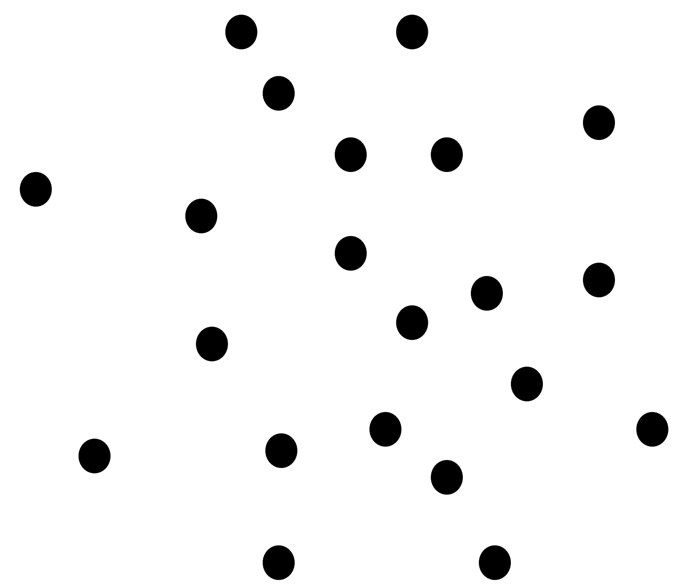
On numbers

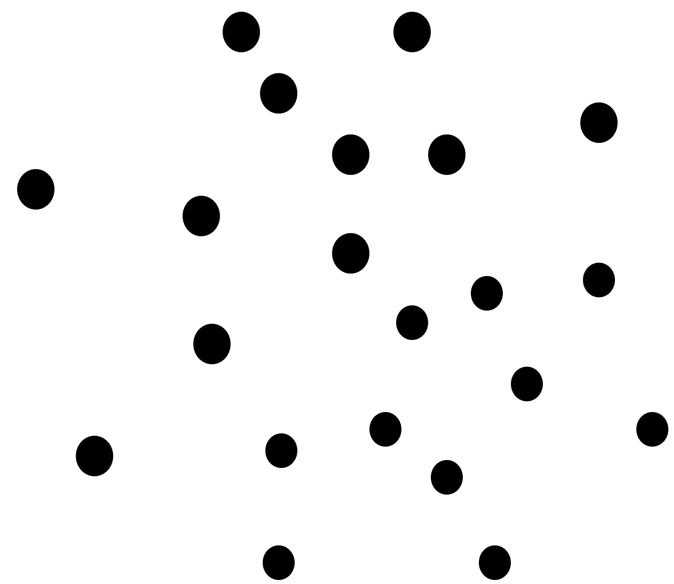
A pearl seminar 8.3 2013

Mats Ehrnström

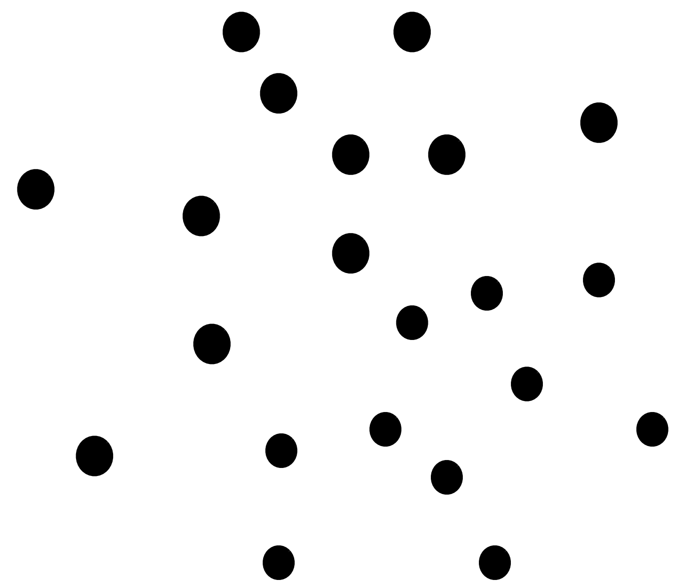




We actually need to *count*.



We actually need to *count*.



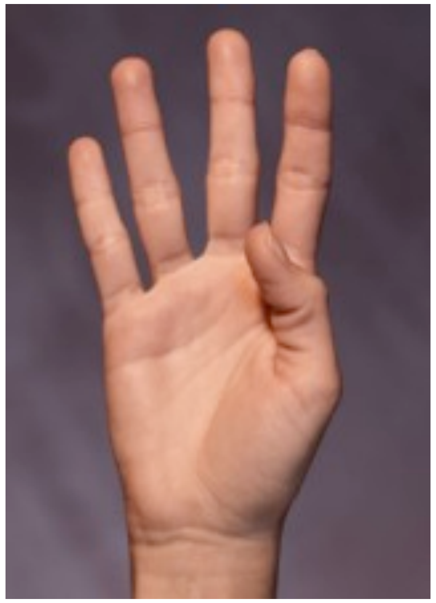
We actually need to *count*.

But already know, *without having counted all*,
many of us feel we have counted more than half.

Quantity perception—the ability to discern ratios—is innate in human beings and many animals. Infants typically succed below a 4:5 ratio (newborns lower).

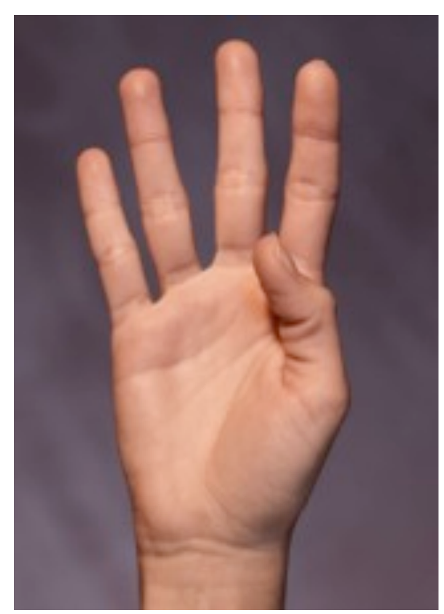
Quantity perception—the ability to discern ratios—is innate in human beings and many animals. Infants typically succed below a 4:5 ratio (newborns lower).

Subitizing is the ability to *immediately* and correctly determine low quantities (typically <5) without counting them. This universal ability does not limit itself to visual objects, but includes sounds and tactile perception.

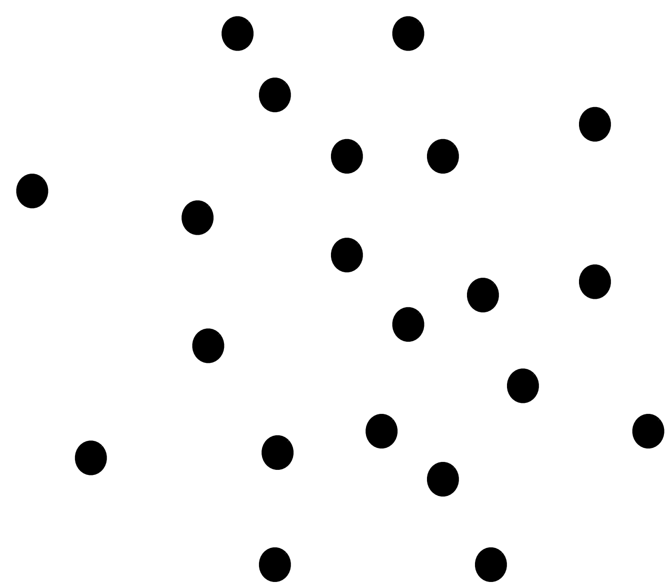


Quantity perception—the ability to discern ratios—is innate in human beings and many animals. Infants typically succed below a 4:5 ratio (newborns lower).

Subitizing is the ability to *immediately* and correctly determine low quantities (typically <5) without counting them. This universal ability does not limit itself to visual objects, but includes sounds and tactile perception.



But man *counts*.



Typically, ordered sets are perceived as larger than unordered sets.

Peano's axioms

But man *counts*.

Peano's axioms

But man *counts*.

A1. 1 is a natural number.

Peano's axioms

But man *counts*.

A1. 1 is a natural number.

A2–5. $n=n$; $m=n \Rightarrow n=m$; $m=n$, $n=k \Rightarrow m=k$;

n natural, $m=n \Rightarrow m$ natural

Peano's axioms

But man *counts*.

A1. 1 is a natural number.

A2-5. n=n; m=n => n=m; m=n, n=k => m=k;

n natural, m=n => m natural

A6. Every natural number n has a successor, S(n).

Peano's axioms

But man *counts*.

A1. 1 is a natural number.

A2–5. $n=n$; $m=n \Rightarrow n=m$; $m=n$, $n=k \Rightarrow m=k$;

n natural, $m=n \Rightarrow m$ natural

A6. Every natural number n has a successor, $S(n)$.

A7. 1 is the successor of no natural number.

Peano's axioms

But man *counts*.

A1. 1 is a natural number.

A2–5. $n=n$; $m=n \Rightarrow n=m$; $m=n$, $n=k \Rightarrow$ $m=k$;

n natural, $m=n \Rightarrow m$ natural

A6. Every natural number n has a successor, $S(n)$.

A7. 1 is the successor of no natural number.

A8. $S(m) = S(n) \Rightarrow m=n$

Peano's axioms

But man *counts*.

A1. 1 is a natural number.

A2-5. $n=n$; $m=n \Rightarrow n=m$; $m=n$, $n=k \Rightarrow m=k$;

n natural, $m=n \Rightarrow m$ natural

A6. Every natural number n has a successor, $S(n)$.

A7. 1 is the successor of no natural number.

A8. $S(m) = S(n) \Rightarrow m=n$

A9. Axiom of induction

Peano's axioms

But man *counts*.

A1. 1 is a natural number.

A2–5. $n=n$; $m=n \Rightarrow n=m$; $m=n,\ n=k \Rightarrow m=k$;

n natural, $m=n \Rightarrow m$ natural

A6. Every natural number n has a successor, $S(n)$.

A7. 1 is the successor of no natural number.

A8. $S(m) = S(n) \Rightarrow m=n$

A9. Axiom of induction

/
/
/
/
/
/
/
/
/
/
/
/
/
/
/

But man *counts*.

The *Lebombo bone*
—the oldest known
mathematical artefact

/
/
/
/
/
/
/
/
/
/
/
/
/
/
/

But man *counts*.

The *Lebombo bone* —the oldest known mathematical artefact

Dated to 35 000 B.C.

Found in the Lebombo Mountains, Swaziland.

Contains 29 (30?) notches carved into baboon bone.

/ / / / / / / / / / / / / / /

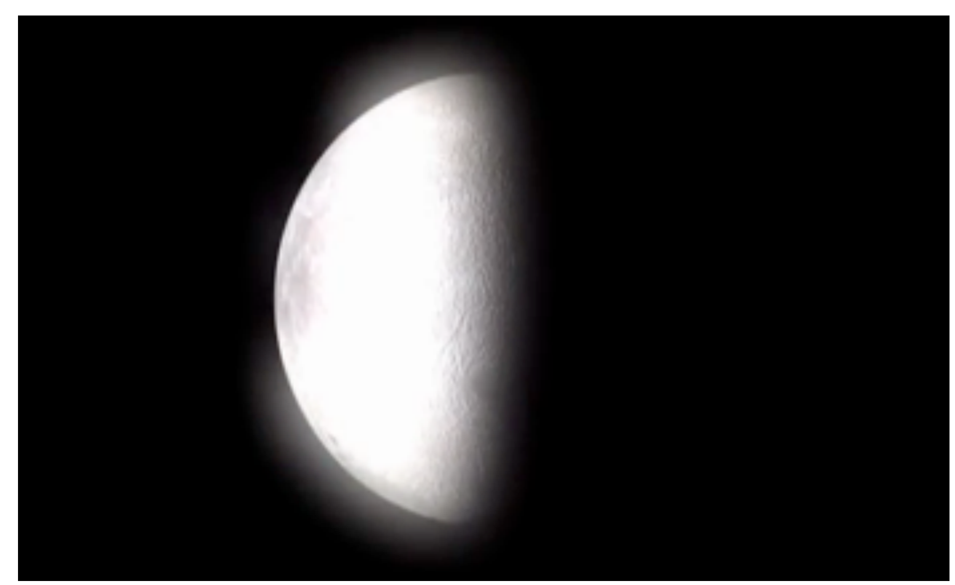
But man *counts*.

The *Lebombo bone*
—the oldest known
mathematical artefact

Dated to 35 000 B.C.

Found in the Lebombo Mountains, Swaziland.

Contains 29 (30?) notches carved into baboon bone.



A lunar cycle is 29.53 days

The second oldest is the *Ishango bone*.

But man *counts*.

/
/
/
/
/
/
/
/
/
/
/
/
/
/
/

The second oldest is the *Ishango bone*.

Older than 20 000 years.

Found near Virunga National Park, Democratic Republic of Congo.

Contains three columns of grouped notches.

But man *counts*.

The second oldest is the *Ishango bone*.

11, 13, 17, 19

But man *counts*.

/
/
/
/
/
/
/
/
/
/
/
/
/
/
/

The second oldest is the *Ishango bone*.

11, 13, 17, 19

A prime quadruplet:
p, p+2 p+6, p+8.

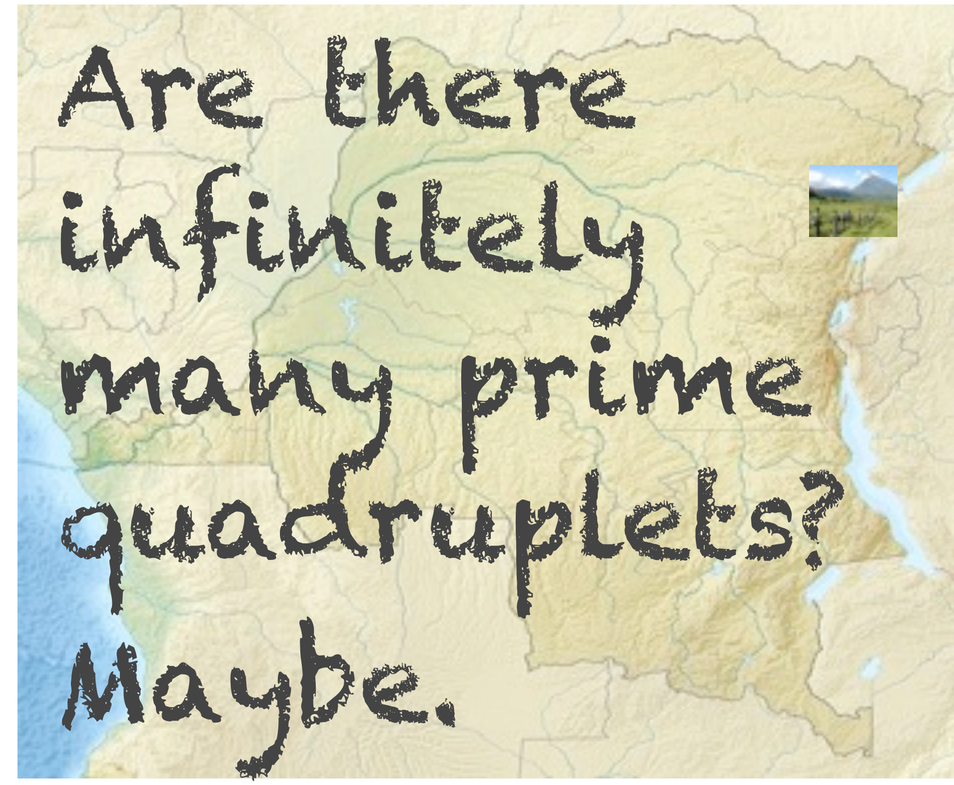
Closest possible grouping for p>3.
Proof: 0,2,4 = 0,2,1 (mod 3)

But man *counts*.

/
/
/
/
/
/
/
/
/
/
/
/
/
/
/

The second oldest is the *Ishango bone*.

But man *counts*.



11, 13, 17, 19

A prime quadruplet:
p, $p+2$ $p+6$, $p+8$.

Closest possible grouping for $p>3$.
Proof: $0,2,4 = 0,2,1 \pmod 3$

But of course, this might all be for the grip…

So we learnt to count,
and we learnt that there is no end of numbers.

So we learnt to count,
and we learnt that there is no end of numbers.

But what is ?

So we learnt to count,
and we learnt that there is no end of numbers.

But what is   ?

And how do we separate III IIII from III   IIII?

So we learnt to count,
and we learnt that there is no end of numbers.

But what is ?

And how do we separate III IIII from III IIII?

Or III III from III III?

We need a zero. And we need some signs.

$$- \qquad 0 \qquad +$$

We need a zero. And we need some signs.

But are they really signs?

$$- \qquad 0 \qquad +$$

Of these, $+$ is the easiest.

Addition is the mapping N x N -> N,
such that

A'1: $m+1 = S(m)$,

A'2: $m+S(n) = S(m+n)$.

This reduces addition to counting.

$$- \qquad 0 \qquad +$$

0 is much worse.

$$- \qquad 0 \qquad +$$

0 is much worse.

It is a *placeholder*, a representation of *nothing* (i.e., of 'no thing'), and an *additive identity*.

$$- \qquad 0 \qquad +$$

0 is much worse.

It is a *placeholder*, a representation of *nothing* (i.e., of 'no thing'), and an *additive identity*.

Placeholder:  

$$- \qquad 0 \qquad +$$

0 is much worse.

It is a *placeholder*, a representation of *nothing* (i.e., of ’no thing’), and an *additive identity*.

Placeholder:

1 4

1 0 4

$$- \qquad 0 \qquad +$$

0 is much worse.

It is a *placeholder*, a representation of *nothing* (i.e., of ’no thing’), and an *additive identity*.

Placeholder:

$1 \quad 4$

$1 \quad 0 \quad 4$

Nothing: modern (Zermelo–Fraenkel) set theory introduces zero as the empty set:

$$0 = \{\}$$

0 is much worse.

It is a *placeholder*, a representation of *nothing* (i.e., of ’no thing’), and an *additive identity*.

Placeholder:

1 4

1 0 4

Nothing: modern (Zermelo–Fraenkel) set theory introduces zero as the empty set:

$$0 = \{\}$$

$$1 = 0 \cup \{0\} = \{\} \cup \{\{\}\} = \{\{\}\} = \{0\}.$$

0 is much worse.

It is a *placeholder*, a representation of *nothing* (i.e., of 'no thing'), and an *additive identity*.

Placeholder:

$1 \quad 4$

$1 \quad 0 \quad 4$

Nothing: modern (Zermelo–Fraenkel) set theory introduces zero as the empty set:

$$0 = \{\}$$

$$1 = 0 \cup \{0\} = \{\} \cup \{\{\}\} = \{\{\}\} = \{0\}.$$

$$2 = 1 \cup \{1\} = \{\{\}\} \cup \{\ \{\{\}\}\ \} = \{\ \{\}, \{\{\}\}\ \} = \{0, 1\}.$$

0 is much worse.

It is a *placeholder*, a representation of *nothing* (i.e., of ’no thing’), and an *additive identity*.

Placeholder:

1 4

1 0 4

Nothing: modern (Zermelo–Fraenkel) set theory introduces zero as the empty set:

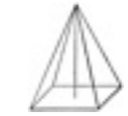
$$0 = \{\}$$

$$1 = 0 \cup \{0\} = \{\} \cup \{\{\}\} = \{\{\}\} = \{0\}.$$

$$2 = 1 \cup \{1\} = \{\{\}\} \cup \{\,\{\{\}\}\,\} = \{\,\{\}, \{\{\}\}\,\} = \{0, 1\}.$$

$$3 = 2 \cup \{2\} = \ldots = \{\,\{\}, \{\{\}\}, \{\{\{\}\}\}\,\} = \{0, 1, 2\}.$$

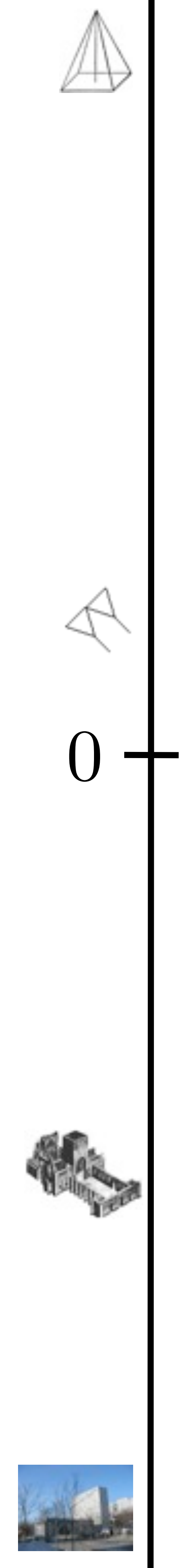
The strongest argument for that the notion of 0 is difficult, is probably history itself.



0

The strongest argument for that the notion of 0 is difficult, is probably history itself.

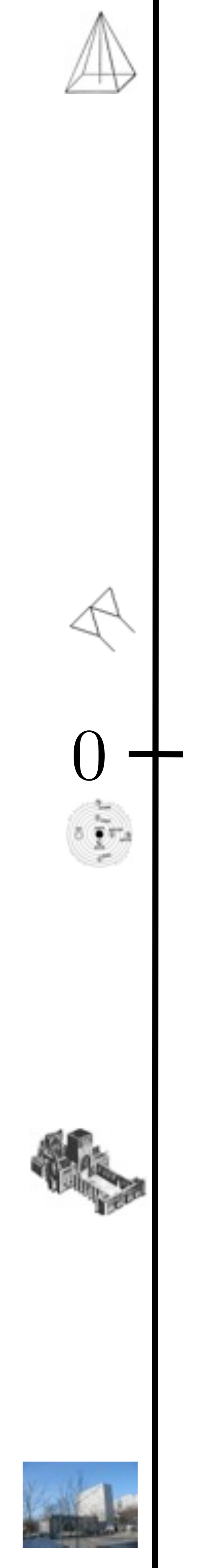
Zero as a placeholder appears only 300 BC (~2000 years later than the decimal system and the first calculations of π.)



The strongest argument for that the notion of 0 is difficult, is probably history itself.

Zero as a placeholder appears only 300 BC (~2000 years later than the decimal system and the first calculations of π.)

Differing vague notions of zero as a symbol for nothing appearing from around 100 AD and onwards (e.g. in Ptolemy's Almagest).

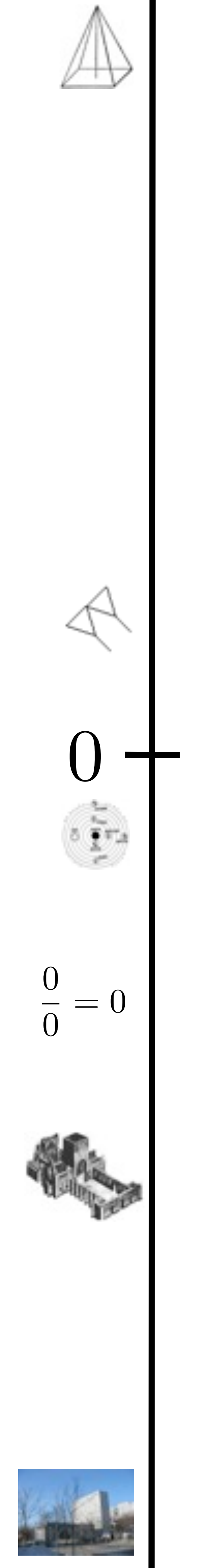


The strongest argument for that the notion of 0 is difficult, is probably history itself.

Zero as a placeholder appears only 300 BC (~2000 years later than the decimal system and the first calculations of π.)

Differing vague notions of zero as a symbol for nothing appearing from around 100 AD and onwards (e.g. in Ptolemy’s Almagest).

The use of zero *as a number* is generally credited to the indians. In 628 AD Brahmagupta published arithmetic rules for zero. The modern place–value decimal system also appeared in India at that time.



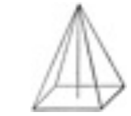
The strongest argument for that the notion of 0 is difficult, is probably history itself.

Zero as a placeholder appears only 300 BC (~2000 years later than the decimal system and the first calculations of π.)

Differing vague notions of zero as a symbol for nothing appearing from around 100 AD and onwards (e.g. in Ptolemy’s Almagest).

The use of zero *as a number* is generally credited to the indians. In 628 AD Brahmagupta published arithmetic rules for zero. The modern place–value decimal system also appeared in India at that time.

Persian *sifr* (meaning ‘empty’) for zero in 976 AD.




0

$\frac{0}{0} = 0$


Still $-$ is worst of them all.

It is the sign of both a *binary* and a *unary operation*, a *part of the sign* for additive inverses, and a notion that *lacks (primitive) real world representations*.

Still $-$ is worst of them all.

It is the sign of both a *binary* and a *unary operation*, a *part of the sign* for additive inverses, and a notion that *lacks (primitive) real world representations*.

– is a mapping, say, Z X Z -> Z (as in 0-3).

Still $-$ is worst of them all.

It is the sign of both a *binary* and a *unary operation*, a *part of the sign* for additive inverses, and a notion that *lacks (primitive) real world representations*.

$-$ is a mapping, say, Z X Z -> Z (as in 0-3).

$-$ is a mapping Z -> Z (as in -3).

Still $-$ is worst of them all.

It is the sign of both a *binary* and a *unary operation*, a *part of the sign* for additive inverses, and a notion that *lacks (primitive) real world representations*.

- is a mapping, say, Z X Z -> Z (as in 0-3).

- is a mapping Z -> Z (as in -3).

- is part of the signs for Z_ (as in -3).

Still —— is worst of them all.

It is the sign of both a *binary* and a *unary operation*, a *part of the sign* for additive inverses, and a notion that *lacks (primitive) real world representations*.

- is a mapping, say, Z X Z -> Z (as in 0-3).

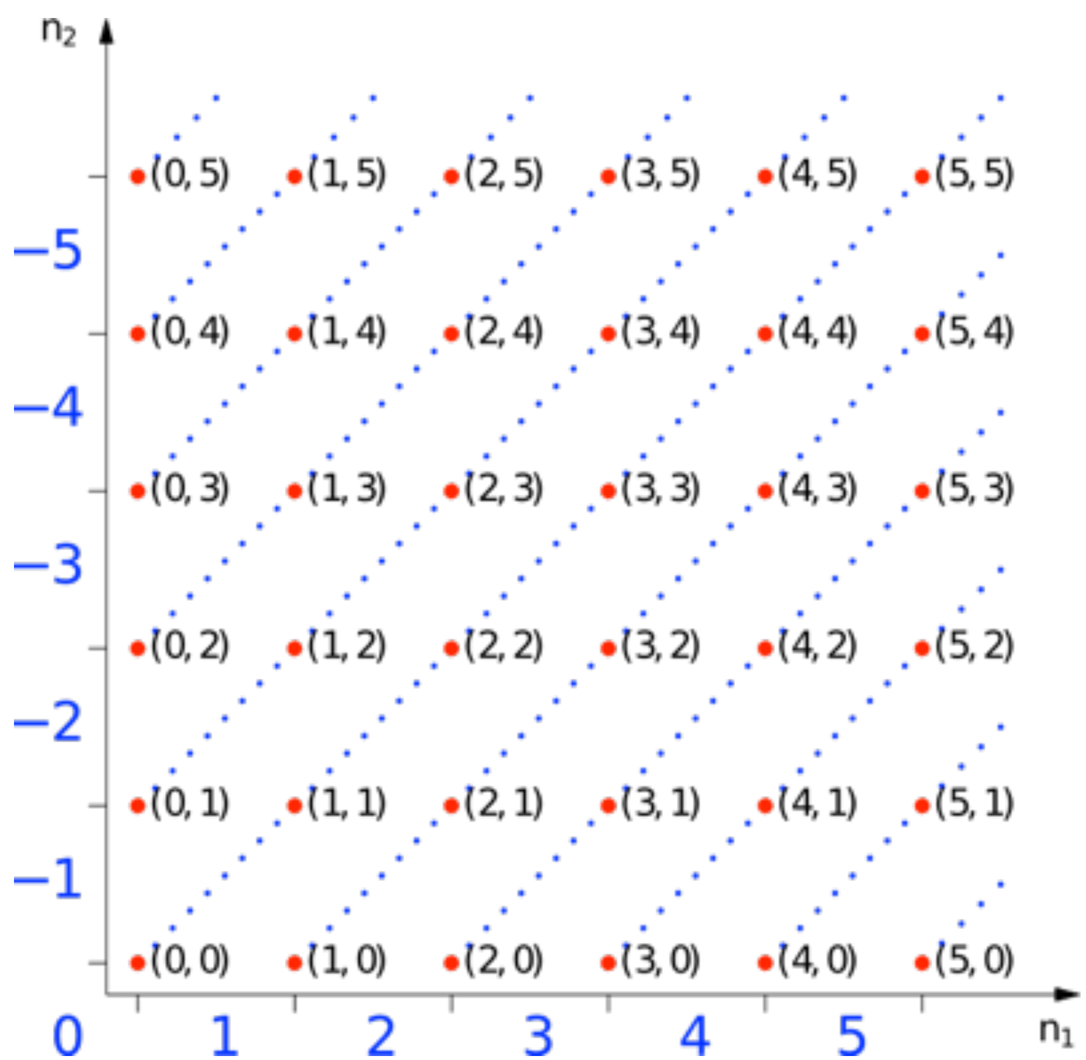
- is a mapping Z -> Z (as in -3).

- is part of the signs for Z_ (as in -3).

And what is (0) - (3) = -(3) = (-3) anyway?

And what is $(0) - (3) = -(3) = (-3)$ anyway?

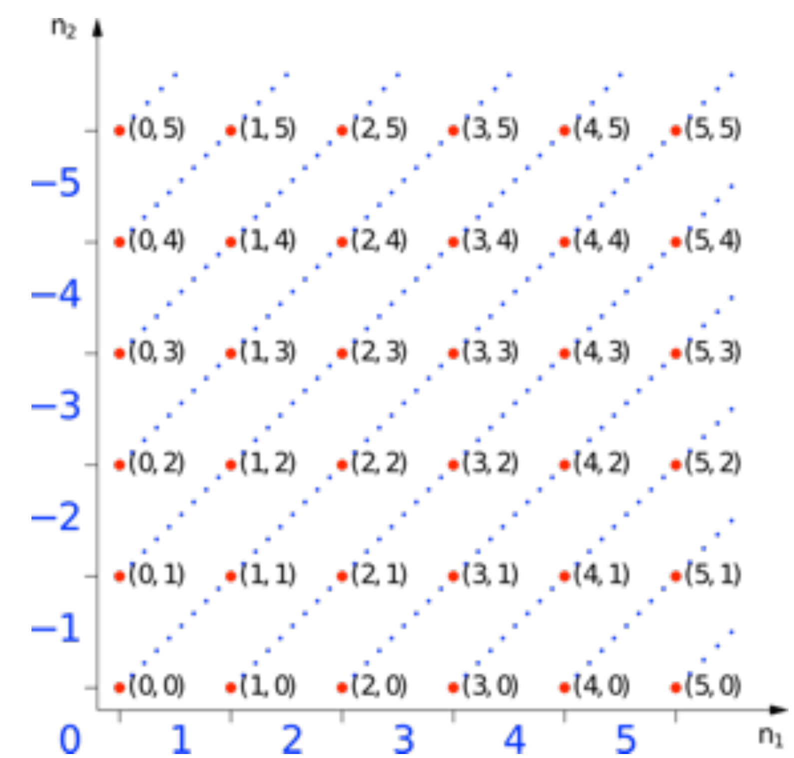
-3 is the equivalence class of all pairs of natural numbers (m,n) of the form $(0,3)$, $(1,4)$, $(2,5)$, ...



And what is (0) - (3) = -(3) = (-3) anyway?

-3 is the equivalence class of all pairs of natural numbers (m,n) of the form (0,3), (1,4), (2,5), ...

$$[(m,n)] + [(k,l)] = [(m+k, n+l)]$$

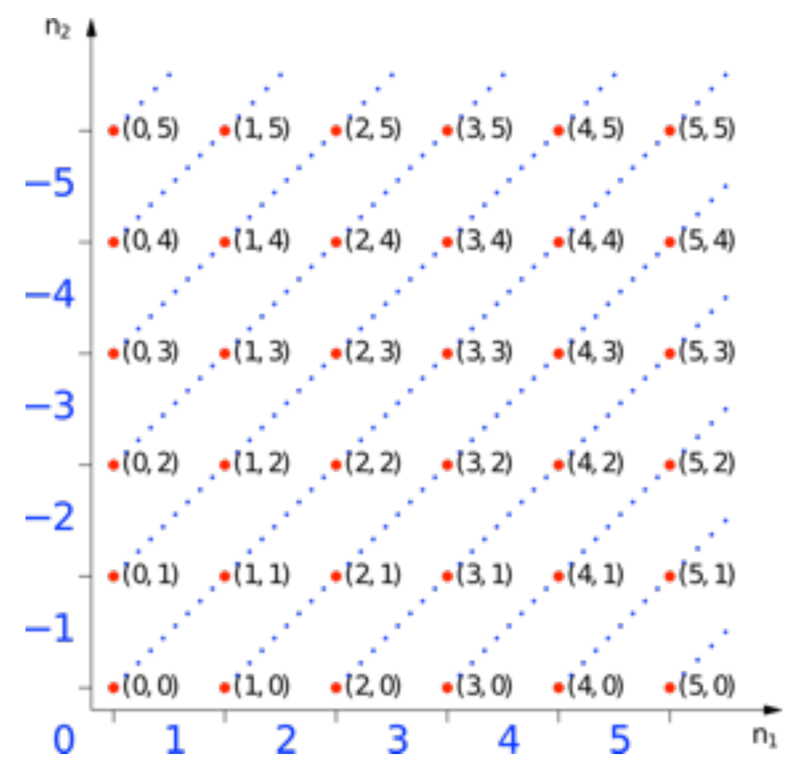


And what is (0) - (3) = -(3) = (-3) anyway?

-3 is the equivalence class of all pairs of natural numbers (m,n) of the form (0,3), (1,4), (2,5), ...

$$[(m,n)] + [(k,l)] = [(m+k, n+l)]$$

$$-[(m,n)] = [(n,m)]$$



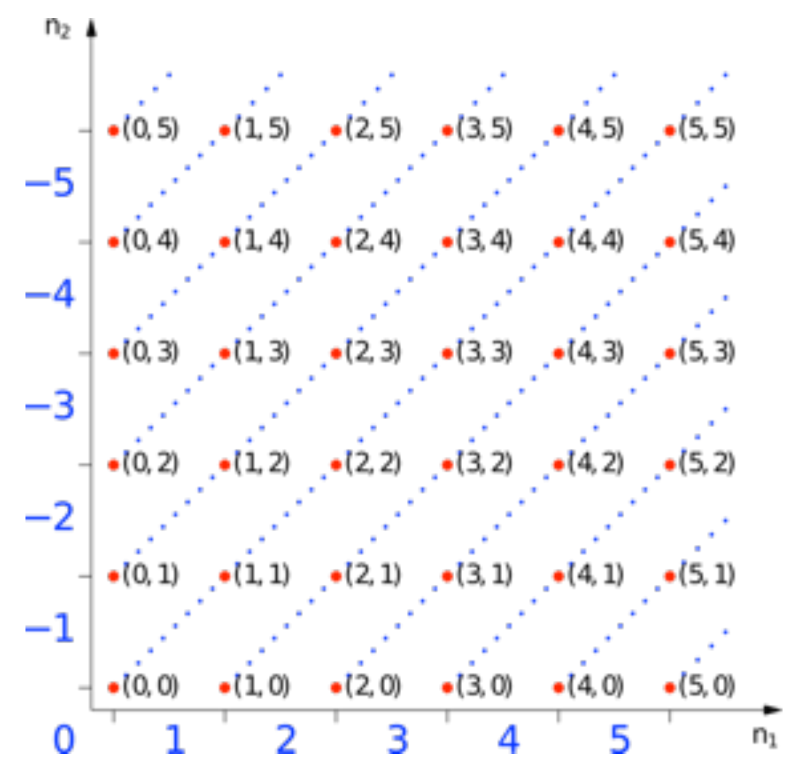
And what is (0) - (3) = -(3) = (-3) anyway?

-3 is the equivalence class of all pairs of natural numbers (m,n) of the form (0,3), (1,4), (2,5), ...

$$[(m,n)] + [(k,l)] = [(m+k, n+l)]$$

$$-[(m,n)] = [(n,m)]$$

$$[(m,n)] - [(k,l)] = [(m,n)] + [(l,k)]$$



And what is (0) - (3) = -(3) = (-3) anyway?

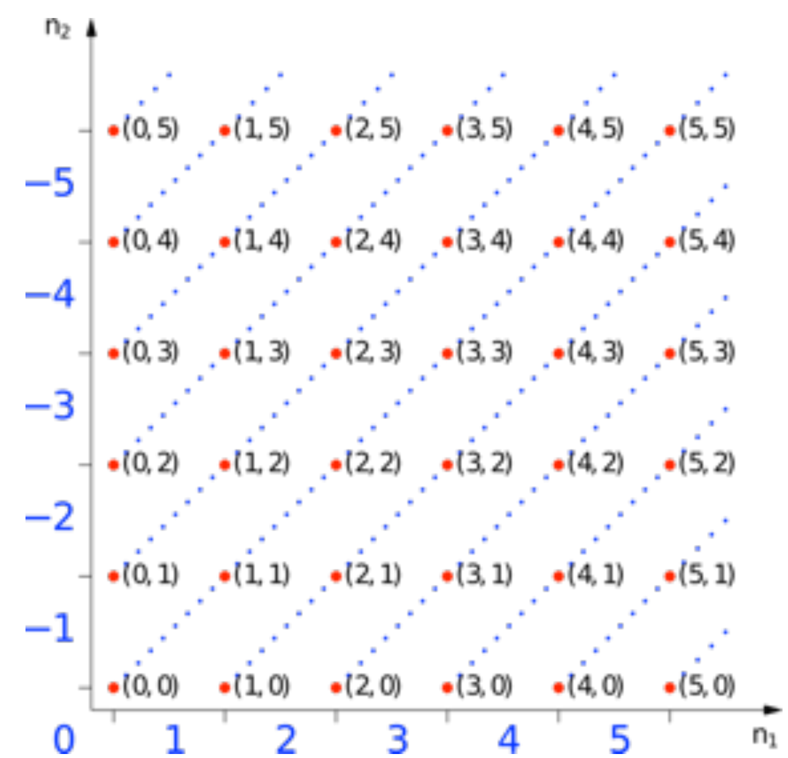
-3 is the equivalence class of all pairs of natural numbers (m,n) of the form (0,3), (1,4), (2,5), ...

$$[(m,n)] + [(k,l)] = [(m+k, n+l)]$$

$$-[(m,n)] = [(n,m)]$$

$$[(m,n)] - [(k,l)] = [(m,n)] + [(l,k)]$$

$$[(m,n)] \times [(k,l)] = [(mk+nl, ml+nk)]$$



And what is (0) - (3) = -(3) = (-3) anyway?

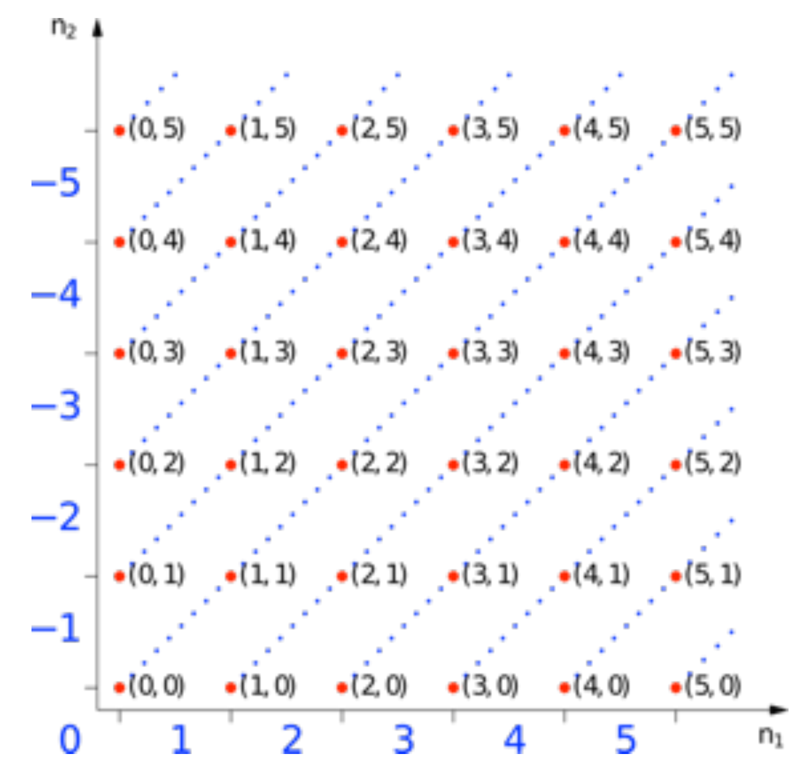
-3 is the equivalence class of all pairs of natural numbers (m,n) of the form (0,3), (1,4), (2,5), ...

$$[(m,n)] + [(k,l)] = [(m+k, n+l)]$$

$$-[(m,n)] = [(n,m)]$$

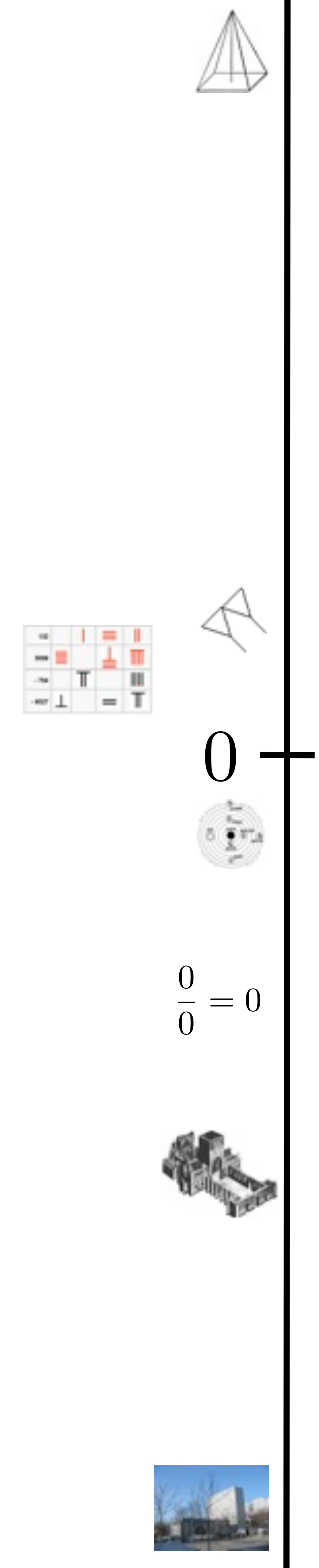
$$[(m,n)] - [(k,l)] = [(m,n)] + [(l,k)]$$

$$[(m,n)] \times [(k,l)] = [(mk+nl, ml+nk)]$$



But this took a long time...

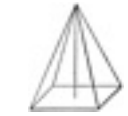
The first recorded use of negative numbers is Chinese, from around 200 BC.



The first recorded use of negative numbers is Chinese, from around 200 BC.

But:

With almost no exception, negative numbers has historically always been introduced and understood as debts or financial deficits.



0

$$\frac{0}{0} = 0$$

The first recorded use of negative numbers is Chinese, from around 200 BC.

But:

With almost no exception, negative numbers has historically always been introduced and understood as debts or financial deficits.

There has been confusion in the separation of the operation and the elements.

0

$\frac{0}{0} = 0$

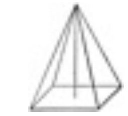
The first recorded use of negative numbers is Chinese, from around 200 BC.

But:

With almost no exception, negative numbers has historically always been introduced and understood as debts or financial deficits.

There has been confusion in the separation of the operation and the elements.

It is not clear when the number line (with negative numbers) was introduced, or if it is even appropriate for man's (or children's) understanding of numbers and quantities.



0

$\frac{0}{0} = 0$

The first recorded use of negative numbers is Chinese, from around 200 BC.

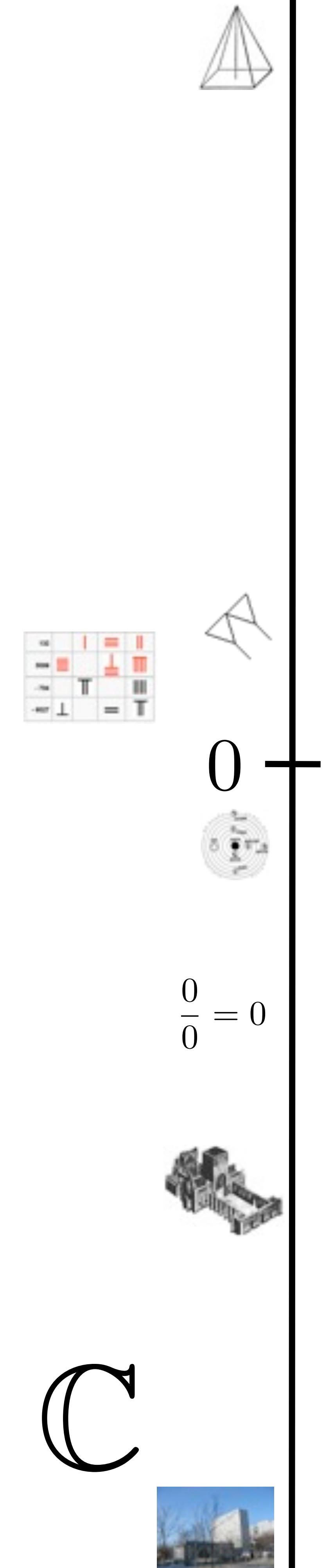
But:

With almost no exception, negative numbers has historically always been introduced and understood as debts or financial deficits.

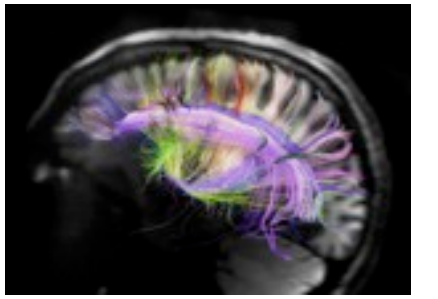
There has been confusion in the separation of the operation and the elements.

It is not clear when the number line (with negative numbers) was introduced, or if it is even appropriate for man's (or children's) understanding of numbers and quantities.

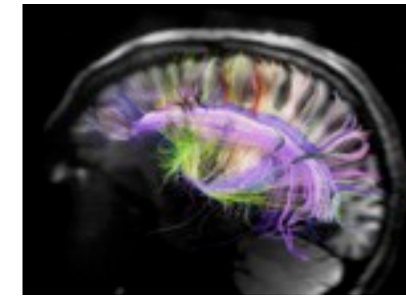
Negative integers (*as numbers*) were generally accepted by mathematicians only as the same time as the complex numbers.



How do humans store or represent numbers and quantities?



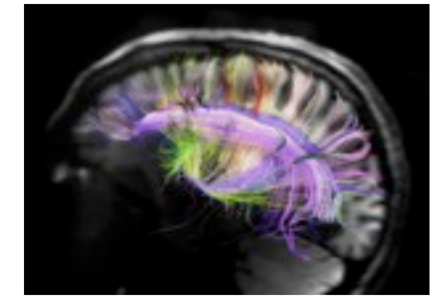
How do humans store or represent numbers and quantities?



Most of neuroscience is due to human disaster and tragedy.

Phineas Gage (1823–1860)

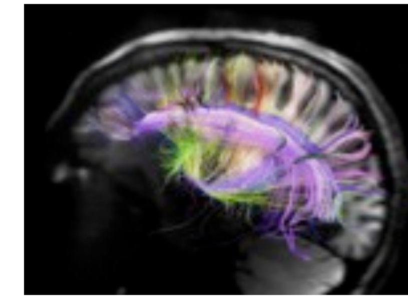
How do humans store or represent numbers and quantities?



Most of neuroscience is due to human disaster and tragedy.

Phineas Gage (1823–1860)

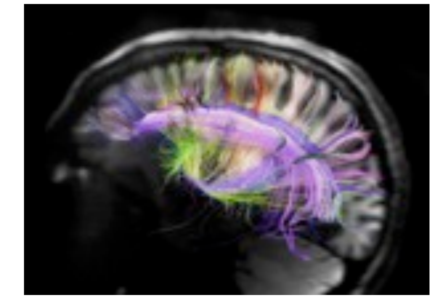
How do humans store or represent numbers and quantities?



Most of neuroscience is due to human disaster and tragedy.

Phineas Gage (1823–1860)

How do humans store or represent numbers and quantities?

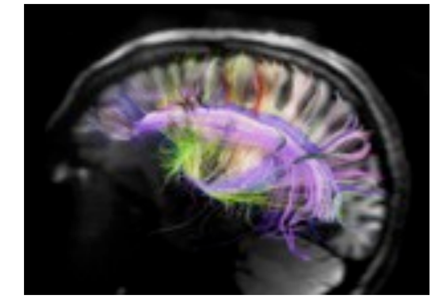


Clear:

– there is a separate mental structure (not necessarily localised) for handling numbers.

Signora Gaddi:

How do humans store or represent numbers and quantities?



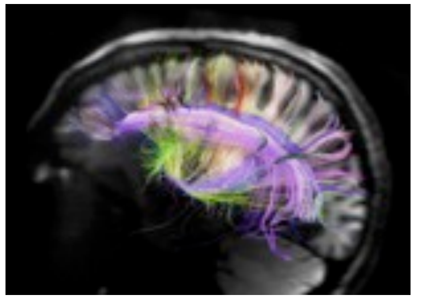
Clear:

– there is a separate mental structure (not necessarily localised) for handling numbers.

Signora Gaddi:

– the ability to handle numbers is separate from the ability to handle logical propositions.

How do humans store or represent numbers and quantities?



Clear:

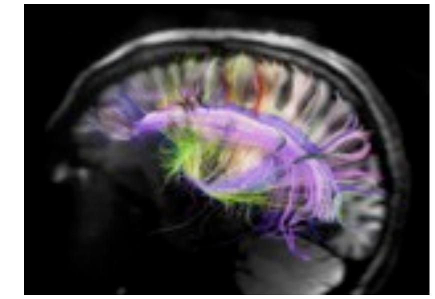
– there is a separate mental structure (not necessarily localised) for handling numbers.

Signora Gaddi:

– the ability to handle numbers is separate from the ability to handle logical propositions.

– handling large numbers is a *mnemonic* feat. It can be trained, but in some persons is due to *synesthesia*.

How do humans store or represent numbers and quantities?



Clear:

– there is a separate mental structure (not necessarily localised) for handling numbers.

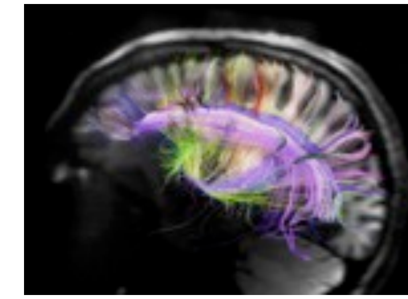
Signora Gaddi:

– the ability to handle numbers is separate from the ability to handle logical propositions.

Daniel Tammet:

– handling large numbers is a *mnemonic* feat. It can be trained, but in some persons is due to *synesthesia*.

How do humans store or represent numbers and quantities?



To Tammet every integer 0–10 000 has a specific form, touch, smell, or other characteristic.

Signora Gaddi:

Daniel Tammet:

– handling large numbers is a *mnemonic* feat. It can be trained, but in some persons is due to *synesthesia*.

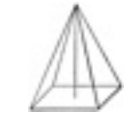
$$\infty$$

0

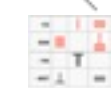
$\frac{0}{0} = 0$

$\mathbb{C}$

The concept of infinity (*as such*) is evidence at least from 300 B.C., both in European and Indian writings.



∞



0

$\frac{0}{0} = 0$

$\mathbb{C}$

The concept of infinity (*as such*) is evidence at least from 300 B.C., both in European and Indian writings.

The concept of infinity (*as innumerability*) is probably innate in humans, and present as soon as a child can count.

The concept of infinity (*as such*) is evidence at least from 300 B.C., both in European and Indian writings.

The concept of infinity (*as innumerability*) is probably innate in humans, and present as soon as a child can count.

Even though, in much of mathematics, the symbol ∞ is not actually used for any object, but as a shorthand for limits, the concept of *infinite sets* (and processes) is unavoidable.

The concept of infinity (*as such*) is evidence at least from 300 B.C., both in European and Indian writings.

The concept of infinity (*as innumerability*) is probably innate in humans, and present as soon as a child can count.

Even though, in much of mathematics, the symbol ∞ is not actually used for any object, but as a shorthand for limits, the concept of *infinite sets* (and processes) is unavoidable.

—which leads us to Cantor, Russell, Gödel, Cohen, and the continuum hypothesis…

In the late 19th century several mathematicians started rejecting the idea (from Galileo and Euclid) that the whole must be larger than any one part of it.

In the late 19th century several mathematicians started rejecting the idea (from Galileo and Euclid) that the whole must be larger than any one part of it.

In the late 19th century several mathematicians started rejecting the idea (from Galileo and Euclid) that the whole must be larger than any one part of it.

Most famously, Cantor, using the concept of cardinality, that

(i) $|A| = |B| \iff A$ is a bijective image of B,

(ii) Each cardinality has a representative set,

showed that the powerset (set of all subsets) of any set always has a greater cardinality than the set itself.

In the late 19th century several mathematicians started rejecting the idea (from Galileo and Euclid) that the whole must be larger than any one part of it.

Most famously, Cantor, using the concept of cardinality, that

(i) $|A| = |B|$ <=> A is a bijective image of B,

(ii) Each cardinality has a representative set,

showed that the powerset (set of all subsets) of any set always has a greater cardinality than the set itself.

$$|\mathbb{N}| = \aleph_0 < \aleph_1 \text{ (size of the set of all countable ordinals)}$$

$$|\mathbb{R}| = 2^{\aleph_0} \text{ (size of the powerset of } \mathbb{N})$$

The continuum hypothesis: $\aleph_1 = 2^{\aleph_0}$?

The continuum hypothesis: $\aleph_1 = 2^{\aleph_0}$?

But a number of obstacles arose:

The continuum hypothesis: $\aleph_1 = 2^{\aleph_0}$?

But a number of obstacles arose:

1901 Russell's paradox $R = \{x \notin x\} \implies [R \in R \iff R \notin R]$

The continuum hypothesis: $\aleph_1 = 2^{\aleph_0}$?

But a number of obstacles arose:

1901 Russell's paradox $R = \{x \notin x\} \implies [R \in R \iff R \notin R]$

Saved by Zermelo–Fraenkel set theory (ZFC).

The continuum hypothesis: $\aleph_1 = 2^{\aleph_0}$?

But a number of obstacles arose:

1901 Russell's paradox $R = \{x \notin x\} \implies [R \in R \iff R \notin R]$

Saved by Zermelo–Fraenkel set theory (ZFC).

1931 Gödel's first incompleteness theorem:

A formal system with an inductively enumerable set of axioms capable of expressing fundamental (Peano) arithmetic cannot be both complete and consistent.

The continuum hypothesis: $\aleph_1 = 2^{\aleph_0}$?

But a number of obstacles arose:

1901 Russell's paradox $R = \{x \notin x\} \implies [R \in R \iff R \notin R]$

Saved by Zermelo–Fraenkel set theory (ZFC).

1931 Gödel's first incompleteness theorem:

A formal system with an inductively enumerable set of axioms capable of expressing fundamental (Peano) arithmetic cannot be both complete and consistent.

1940 Gödel shows that the continuum hypothesis cannot be disproved by means of ZFC.

The continuum hypothesis: $\aleph_1 = 2^{\aleph_0}$?

But a number of obstacles arose:

1901 Russell's paradox $R = \{x \notin x\} \implies [R \in R \iff R \notin R]$

Saved by Zermelo–Fraenkel set theory (ZFC).

1931 Gödel's first incompleteness theorem:

A formal system with an inductively enumerable set of axioms capable of expressing fundamental (Peano) arithmetic cannot be both complete and consistent.

1940 Gödel shows that the continuum hypothesis cannot be disproved by means of ZFC.

1963 Cohen shows that the continuum hypothesis cannot be proved by means of ZFC.

One can assume, *or not assume*, the existence of a set of cardinality strictly between that of the natural numbers and that of the reals, and do mathematics with or without this assumption — *and there is no logical argument against it.*

One can assume, *or not assume*, the existence of a set of cardinality strictly between that of the natural numbers and that of the reals, and do mathematics with or without this assumption — *and there is no logical argument against it.*

Still

– Some believe the continuum hypothesis is false; others believe it is true.

One can assume, *or not assume*, the existence of a set of cardinality strictly between that of the natural numbers and that of the reals, and do mathematics with or without this assumption — *and there is no logical argument against it.*

Still

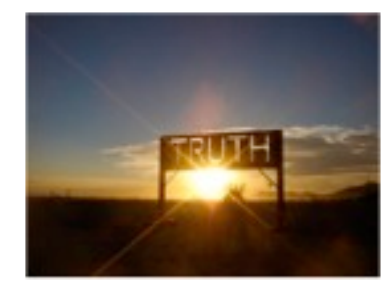
– Some believe the continuum hypothesis is false; others believe it is true.

– Some believe ZFC lacks some important axiom.

One can assume, *or not assume*, the existence of a set of cardinality strictly between that of the natural numbers and that of the reals, and do mathematics with or without this assumption — *and there is no logical argument against it.*

Still

– Some believe the continuum hypothesis is false; others believe it is true.



– Some believe ZFC lacks some important axiom.

– And some think it doesn't have to do with mathematics.

One can assume, *or not assume*, the existence of a set of cardinality strictly between that of the natural numbers and that of the reals, and do mathematics with or without this assumption — *and there is no logical argument against it.*

Still

– Some believe the continuum hypothesis is false; others believe it is true.

– Some believe ZFC lacks some important axiom.

– And some think it doesn't have to do with mathematics.

But possibly more disturbing is the fact that this might be just one of many propositions that 'doesn't have to do with mathematics'.

To put it short:

Today’s concepts of numbers and arithmetics have taken a long time to understand and develop.

To put it short:

Today’s concepts of numbers and arithmetics have taken a long time to understand and develop.

Our everyday use and understanding of these might not always correspond to either their historical or current mathematical meaning.

To put it short:

Today’s concepts of numbers and arithmetics have taken a long time to understand and develop.

Our everyday use and understanding of these might not always correspond to either their historical or current mathematical meaning.

Although the level of abstraction has no end, there is a clear and universal level of when we need to make an effort, and when not

To put it short:

Today's concepts of numbers and arithmetics have taken a long time to understand and develop.

Our everyday use and understanding of these might not always correspond to either their historical or current mathematical meaning.

Although the level of abstraction has no end, there is a clear and universal level of when we need to make an effort, and when not — and that is by 4.